

Steady Non-Isothermal sPTT Pipe Flow: Energy-Source Equivalence and a Verified Chebyshev Continuation Study

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Abstract: Steady shear flow is a useful verification setting for non-isothermal viscoelastic computations because mechanical stress power and polymer relaxation heating can become exactly equivalent. This work establishes that equivalence for fully developed circular-pipe flow of a linear simplified Phan-Thien–Tanner (sPTT) fluid with radially varying viscosity and relaxation time, and evaluates a wall-cooled dissipation–conduction equilibrium. A prescribed wall temperature closes the thermal problem, while the outward wall heat flux is calculated from the solution. The stress field is eliminated analytically and the squared-radius coordinate converts the cylindrical diffusion operator into a regular polynomial collocation problem. A Chebyshev–Lobatto solver, two analytical benchmark families, and an independent adaptive boundary-value solver provide complementary verification. Eighty parameter combinations span the pressure-scaled Nahme number $0.01 \leq Ng \leq 0.20$, the combined sPTT parameter $0 \leq \chi = \epsilon We_G^2 \leq 0.04$, and the relaxation-to-viscosity temperature-sensitivity ratio $0.5 \leq a \leq 2$. The largest relative wall-energy-balance discrepancy is 2.8×10^{-10} , and six independent-solver comparisons give a maximum relative Nusselt-number difference of 3.2×10^{-10} . Separate evaluation of five energy partitions changes the nodal temperature by at most 1.4×10^{-14} , consistent with the exact identity. Centerline-parameter continuation locates turning points on the computed steady branches: at $\chi = 0.04$, the turning-point Nahme number increases from 0.226413 at $a = 0.5$ to 0.452037 at $a = 2$. These are branch-existence diagnostics, without an inference of dynamic stability. The resulting formulation provides a reproducible verification problem and a quantified account of how property sensitivity affects steady thermal feedback.

Keywords: simplified Phan-Thien–Tanner fluid, viscous heating, temperature-dependent rheology, Chebyshev collocation, energy balance, numerical continuation

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1. Introduction

The Phan-Thien–Tanner constitutive equation provides a network-based representation of polymeric extra stress [1]. Its simplified, zero-slip form permits analytical reduction in fully developed shear flows. Oliveira and Pinho derived channel and pipe solutions for both linear and exponential stress functions [2]. The availability of exact stress and velocity fields makes these geometries particularly useful for verifying coupled numerical calculations.

Thermal boundary conditions determine which heat-transfer benchmark is relevant. Pinho and Oliveira analysed constant-flux pipe and channel convection, including mechanical heating [3]. Coelho, Pinho, and Oliveira subsequently considered constant-wall-temperature limits, including the equilibrium of radial conduction and viscous dissipation [4]. A numerical result for one thermal setting cannot be validated against the limiting Nusselt number of another setting simply because both use a circular pipe.

Temperature-dependent rheology adds feedback between heat generation, material properties, and flow rate. Nóbrega et al. evaluated variable-property corrections in viscoelastic duct flows [5], while Hashemabadi and Mirnajafizadeh employed exponential property laws in Couette flow and thin annuli [6]. These studies establish the relevance of

thermal sensitivity but do not remove the need to specify the pressure or flow-rate control and the thermal state for every parameter comparison.

Energy transport also requires a distinction between stress work, elastic storage, and relaxation. Peters and Baaijens developed a thermodynamic treatment of non-isothermal viscoelastic flow [7]. Rodrigues, Vide, Afonso, and Pinho extended that analysis to material properties that vary with temperature and time [8]. Their framework supplies the context for the reduced energy-source expression examined here. In a steady parallel flow, the constitutive balance can make that source independent of its partition parameter. This equality is a property of the restricted flow setting; it does not justify replacing the full thermodynamic equations in an arbitrary transient or developing flow.

The limited sensitivity to energy partition in fully developed shear has also been discussed in the non-isothermal computational literature. Fernandes presents an implicit log-conformation formulation and notes that steady shear can have no internal-energy storage rate [9]. The present work uses this established distinction as a verification requirement and focuses its numerical analysis on the property-sensitivity dependence and continuation of a precisely defined wall-cooled state.

The present study evaluates a fully developed, wall-cooled equilibrium with prescribed G and T_w . Each streamline remains at a fixed radial location and temperature. This choice produces a closed radial thermal problem while retaining temperature-dependent viscosity and relaxation time. The wall heat flux follows from the solution. The computational contributions are an explicit stress elimination, a squared-radius Chebyshev formulation, verification against exact linear and nonlinear solutions, independent discretization checks, and numerical continuation of temperature-sensitive steady branches. The energy-source equality is stated as a verification identity with established thermodynamic foundations, rather than as a new general theory of viscoelastic heat transfer.

Section 2 defines the model and proves the source identity. Section 3 describes the numerical implementation and error measures. Section 4 presents verification. Section 5 reports the parameter study and continuation. Section 6 sets out the physical and numerical limits of the results.

2. Governing Equations

2.1. Flow Geometry and Kinematics

Consider a straight circular pipe containing a single-mode linear sPTT fluid. The flow is steady, axisymmetric, laminar, and independent of z , with no wall slip or radial velocity:

$$\mathbf{v} = (0, 0, u(r)), \quad T = T(r), \quad \frac{dp}{dz} = -G. \quad (1)$$

Thermal conductivity is constant. The solvent-viscosity contribution, imposed volumetric heating, compressibility, and thermal expansion are omitted. In particular, the coefficient multiplying the pressure-temperature coupling is set to zero as part of the thermodynamic approximation. Incompressibility by itself would not imply $Dp/Dt = 0$ in a pressure-driven flow.

The boundary conditions are

$$u(R) = 0, \quad u'(0) = 0, \quad T(R) = T_w, \quad T'(0) = 0. \quad (2)$$

The wall flux $q_w = -kT'(R)$ is positive for cooling. Here $DT/Dt = u\partial T/\partial z = 0$. Hence material derivatives of $\eta(T)$ and $\lambda(T)$ vanish along a streamline even when those properties vary across the pipe. An axial thermal-entry problem would require a different kinematic and thermal reduction.

2.2. Constitutive Equation: Simplified Phan-Thien–Tanner Model

For the zero-slip linear sPTT model,

$$f\boldsymbol{\tau} + \lambda(T)\overset{\nabla}{\boldsymbol{\tau}} = 2\eta(T)\mathbf{S}, \quad f = 1 + \frac{\epsilon\lambda(T)}{\eta(T)}\text{tr}(\boldsymbol{\tau}), \quad (3)$$

where $\mathbf{S} = (\nabla\mathbf{v} + \nabla\mathbf{v}^T)/2$. The tensor $\boldsymbol{\tau}$ denotes polymeric extra stress and need not be trace-free. Under Eq. (1), $\tau_{rr} = \tau_{\theta\theta} = 0$ and the nonzero balances are

$$f\tau_{rz} = \eta u', \quad f\tau_{zz} = 2\lambda\tau_{rz}u'. \quad (4)$$

The momentum equation and centerline regularity give

$$\frac{1}{r} \frac{d}{dr}(r\tau_{rz}) = -G, \quad \tau_{rz} = -\frac{Gr}{2}. \quad (5)$$

Eliminating u' between the two constitutive balances yields the simpler closed stress expression

$$\tau_{zz} = \frac{2\lambda}{\eta}\tau_{rz}^2, \quad f = 1 + 2\epsilon\left(\frac{\lambda\tau_{rz}}{\eta}\right)^2, \quad u' = \frac{\tau_{rz}}{\eta}f. \quad (6)$$

These formulas retain the local properties explicitly and avoid a subtractive square-root evaluation. They also have a regular Newtonian limit as $\lambda \rightarrow 0$.

2.3. Temperature-Dependent Properties

Let $\beta_\eta > 0$ and $\beta_\lambda \geq 0$. The prescribed exponential laws are

$$\eta(T) = \eta_w e^{-\beta_\eta(T-T_w)}, \quad \lambda(T) = \lambda_w e^{-\beta_\lambda(T-T_w)}. \quad (7)$$

All reference properties are evaluated at the imposed wall temperature. These are phenomenological laws defining the numerical model; no material-specific calibration or universal equivalence to an Arrhenius law is asserted. Their physical application requires measured property curves over the actual temperature interval.

2.4. Energy-Source Equivalence

Under the stated zero-slip, zero-solvent, zero-expansion assumptions, the source obtained from the PTT energy equation in Rodrigues et al. [8] is

$$\mathcal{H}_\phi = \phi\boldsymbol{\tau} : \mathbf{S} + (1 - \phi)\frac{f\text{tr}(\boldsymbol{\tau})}{2\lambda}, \quad 0 \leq \phi \leq 1. \quad (8)$$

Every term has units of power per volume. The relaxation contribution contains a single stress trace. The reduced temperature equation is

$$0 = \frac{k}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \mathcal{H}_\phi. \quad (9)$$

Proposition 1. *For the steady parallel sPTT flow defined by Eqs. (1)–(4), the reduced source satisfies $\mathcal{H}_\phi = \tau_{rz}u'$ for every $\phi \in [0, 1]$ at fixed rheological properties.*

Proof. The two equal shear components of $\mathbf{S}$ give $\boldsymbol{\tau} : \mathbf{S} = \tau_{rz}u'$. Also $\text{tr}(\boldsymbol{\tau}) = \tau_{zz}$. The axial normal-stress balance therefore gives

$$\frac{f\text{tr}(\boldsymbol{\tau})}{2\lambda} = \frac{f\tau_{zz}}{2\lambda} = \tau_{rz}u'. \quad (10)$$

Substitution into Eq. (8) proves the assertion. The Newtonian limit is obtained continuously. $\square$

The identity establishes equal source predictions in this equilibrium despite the different interpretations of the two source contributions. Elastic stress can remain nonzero while its stored energy has no material rate of change. Varying ϕ here is an algebraic verification at fixed property laws, not a calibration of different polymer materials or a construction of distinct free-energy models. Full thermodynamic closure beyond this equilibrium remains governed by the more general constitutive and temperature equations.

2.5. Dimensionless Groups and Reduced Problem

Define

$$U_G = \frac{GR^2}{8\eta_w}, \quad x = \frac{r}{R}, \quad y = x^2, \quad v = \frac{u}{U_G}, \quad \theta = \beta_\eta(T - T_w), \quad (11)$$

and

$$\text{We}_G = \frac{\lambda_w U_G}{R}, \quad \chi = \epsilon \text{We}_G^2, \quad \text{Ng} = \frac{\beta_\eta \eta_w U_G^2}{k}, \quad a = \frac{\beta_\lambda}{\beta_\eta}. \quad (12)$$

Stars on stresses denote division by $\eta_w U_G / R$. Equations (5) and (6) become

$$\tau_{rz}^* = -4x, \quad \tau_{zz}^* = 32\text{We}_G x^2 e^{(1-a)\theta}, \quad f = 1 + 32\chi x^2 e^{2(1-a)\theta}. \quad (13)$$

The velocity gradient and dimensionless mechanical power are

$$\frac{dv}{dx} = -4xe^\theta \left[1 + 32\chi x^2 e^{2(1-a)\theta} \right], \quad (14)$$

$$\mathcal{S}(y, \theta) = 16ye^\theta + 512\chi y^2 e^{(3-2a)\theta}. \quad (15)$$

The exact change of coordinate gives

$$\frac{1}{x} \frac{d}{dx} \left(x \frac{d\theta}{dx} \right) = 4 \left(y \frac{d^2\theta}{dy^2} + \frac{d\theta}{dy} \right). \quad (16)$$

Thus the closed boundary-value problem is

$$4(y\theta_{yy} + \theta_y) + \text{Ng}\mathcal{S}(y, \theta) = 0, \quad \theta(1) = 0, \quad (17)$$

with a regular solution at $y = 0$. Since $\mathcal{S}(0, \theta) = 0$, its limiting equation is $\theta_y(0) = 0$. The velocity then follows from

$$v(y) = 2 \int_y^1 e^{\theta(s)} \left[1 + 32\chi s e^{2(1-a)\theta(s)} \right] ds. \quad (18)$$

An expansion polynomial in y is even in x , so the centerline derivative with respect to x vanishes automatically. The axis is included in the collocation grid and no $1/x$ coefficient is evaluated numerically.

2.6. Heat Transfer and Pressure-Power Balance

The velocity-weighted bulk temperature and mean flow are

$$M = \int_0^1 v(y) dy, \quad \theta_b = \frac{\int_0^1 v(y) \theta(y) dy}{M}, \quad Q = \pi R^2 U_G M. \quad (19)$$

Consequently,

$$q^* = -2\theta_y(1), \quad \text{Nu} = \frac{2q^*}{\theta_b}. \quad (20)$$

This Nu describes the internally heated wall-cooled equilibrium. It is defined for $\text{Ng} > 0$ and has a limiting value as heating tends to zero. Both numerator and denominator vanish at the zero-heating state itself.

Integrating Eq. (17) gives

$$q^* = \frac{\text{Ng}}{2} \int_0^1 \mathcal{S}(y, \theta) dy. \quad (21)$$

Using $\mathcal{S} = -4xv_x$ and integration by parts additionally yields

$$\int_0^1 \mathcal{S} dy = 8M, \quad q^* = 4\text{Ng}M. \quad (22)$$

In dimensional form this is $GQ = 2\pi Rq_w$, the pressure-work input per pipe length equalling the outward wall heat loss. Heat generation remains nonnegative because τ_{rz} and u' have the same sign.

The imposed controls are G and T_w ; the mean velocity is an output. The wall gradient has the exact value

$$-v_x(1) = 4(1 + 32\chi). \quad (23)$$

It is independent of Ng and a at fixed χ . Mean-flow-based groups differ from Eq. (12): $\text{We}_{\text{mean}} = \text{We}_G M$ and $\text{Ng}_{\text{mean}} = \text{Ng} M^2$. A flux-based Brinkman number formed with U_G is an output, $\text{Br}_q = \eta_w U_G^2 / (q_w R) = 1 / (4M)$. It cannot be swept independently while retaining this equilibrium and the prescribed controls.

3. Chebyshev Spectral Collocation Method

3.1. Spatial Discretization

Following polynomial collocation practice [10], use the ascending Chebyshev-Lobatto points

$$y_j = \frac{1 - \cos(j\pi/N)}{2}, \quad j = 0, \dots, N, \quad (24)$$

and the interpolant

$$\theta_N(y) = \sum_{j=0}^N \theta_j \ell_j(y), \quad \ell_j(y_i) = \delta_{ij}. \quad (25)$$

Let $b_j = (-1)^j c_j$, with $c_0 = c_N = 1/2$ and other $c_j = 1$. The differentiation matrix is

$$D_{ij} = \frac{b_j}{b_i(y_i - y_j)} \quad (i \neq j), \quad D_{ii} = -\sum_{j \neq i} D_{ij}. \quad (26)$$

The radial diffusion matrix is $L = 4[\text{diag}(\mathbf{y})D^2 + D]$. Its axis row enforces the regular limiting equation. Its wall row is replaced by $\theta_N(1) = 0$. This explicitly identifies the nodes, basis, scaling, regularity treatment, and boundary enforcement.

3.2. Solution Algorithm

For the non-wall rows, the nonlinear residual and analytical Jacobian are

$$\mathbf{F} = L\boldsymbol{\theta} + \text{Ng}\mathcal{S}, \quad J = L + \text{Ng} \text{diag}(\mathcal{S}_\theta), \quad (27)$$

where

$$\mathcal{S}_\theta = 16ye^\theta + 512\chi(3 - 2a)y^2e^{(3-2a)\theta}. \quad (28)$$

A damped Newton step solves $J\delta\boldsymbol{\theta} = -\mathbf{F}$. A backtracking factor starts at one and is halved until the unscaled infinity norm decreases; at most 30 backtracking attempts and 100 Newton steps are allowed. Failure raises an exception rather than producing a

reported result. A temperature-magnitude bound of 20 is an overflow safeguard in the implementation, and is never reached by the reported solutions.

For the parameter grid, continuation starts from $\theta = 0$ at $\text{Ng} = 0$ and increases Ng in increments no larger than 0.01. This selects the branch connected to the zero-heating state. Its selection is not a dynamic-stability determination. Convergence requires the normalized equation residual

$$\mathcal{R}_N = \frac{\|\mathbf{F}\|_\infty}{1 + \|\text{Ng}\mathbf{S}\|_\infty} < 2 \times 10^{-10}. \quad (29)$$

The wall equation is included in $\mathbf{F}$. Velocity is reconstructed by integrating a Chebyshev interpolant of the integrand in Eq. (18). Bulk quantities use 180-point Gauss–Legendre quadrature in y . Production calculations use $N = 32$.

3.3. Independent Verification and Conservation Checks

An independent adaptive fourth-order collocation solution is obtained using SciPy's boundary-value solver [11,12]. It solves $\theta_y = w$ together with

$$w_y = -\frac{w}{y} - \text{Ng} \left[4e^\theta + 128\chi y e^{(3-2a)\theta} \right]. \quad (30)$$

The singular-matrix interface imposes the regularity condition $w(0) = 0$ without direct division at the axis. The initial mesh has 121 points and the residual tolerance is 2×10^{-9} . Polynomial initial guesses are supplied independently of the spectral solution. Nusselt-number comparisons use the adaptive solution's wall derivative and separately evaluated velocity and bulk integrals with 240-point quadrature.

Conservation is checked using the relative discrepancies between the wall derivative and each right-hand side of Eqs. (21) and (22). The differential residual is also evaluated at the interior quadrature points, rather than only at the collocation nodes. Its normalization is the continuous counterpart of Eq. (29). Errors at these extra points assess interpolation and differentiation defects that a nodal residual alone can miss.

3.4. Centerline-Parameter Continuation

To traverse a steady turning point, prescribe $A = \theta(0)$ and solve for Ng as an additional unknown. The augmented system comprises the $N + 1$ residual equations and $\theta_0 - A = 0$. The extra Jacobian column is $\mathbf{S}$, with zero in the replaced wall row. A normalized residual below 5×10^{-10} is required.

The branch is sampled at 100 centerline values from 0.02 to 2.8. An interior maximum of $\text{Ng}(A)$ is bracketed by neighbouring samples and refined by bounded scalar optimization. The centerline tolerance of the optimizer is 2×10^{-9} . Independent adaptive calculations impose the same centerline value and solve for Ng with an initial 181-point mesh. Thus the reported turning points have both polynomial-order and independent-discretization checks. The method locates the maxima on the traced branches; it does not establish global uniqueness or rule out additional branches outside the traced interval.

4. Validation

4.1. Constant-Property sPTT Limit

Freezing the property ratios at unity gives an auxiliary verification problem whose exact solutions follow directly from Eqs. (17) and (18):

$$v_0(y) = 2(1 - y) + 32\chi(1 - y^2), \quad \theta_0(y) = \text{Ng}h_\chi(y), \quad (31)$$

$$h_\chi(y) = 1 - y^2 + \frac{128\chi}{9}(1 - y^3), \quad M_0 = 1 + \frac{64\chi}{3}. \quad (32)$$

The temperature scaling in this auxiliary calculation is retained as a nominal scale while rheological feedback is disabled. These formulas can also be interpreted as the leading small- Ng terms of the nonlinear problem. With $I_\chi = \int_0^1 v_0 h_\chi dy$,

$$I_\chi = \frac{5}{6} + \frac{448}{15}\chi + \frac{7168}{27}\chi^2, \quad \text{Nu}_0(\chi) = \frac{8M_0^2}{I_\chi}. \quad (33)$$

In particular, $\text{Nu}_0(0) = 48/5 = 9.6$ for the internally heated wall-cooled Newtonian equilibrium. The constant-property computations in Table 1 agree with the exact values to floating-point precision. Since the exact fields are low-degree polynomials, this verifies the operator, boundary conditions, and integral definitions but is insufficient on its own to verify nonlinear convergence.

Table 1. Frozen-property, wall-cooled sPTT verification at $N = 16$. Errors are relative and at the floating-point floor.

χ	Nu (spectral)	Nu (exact)	Relative error
0.0000	9.600000000	9.600000000	5.00×10^{-15}
0.0025	9.757597472	9.757597472	4.01×10^{-15}
0.0100	10.165673512	10.165673512	2.97×10^{-15}
0.0400	11.203150481	11.203150481	5.07×10^{-15}

4.2. Exact Nonlinear Newtonian Benchmark

For $\chi = 0$, the nonlinear equation has the regular family

$$\theta(y) = 2 \log \left(\frac{1+c}{1+cy^2} \right), \quad \text{Ng} = \frac{2c}{(1+c)^2}, \quad c > 0. \quad (34)$$

Substitution into Eq. (17) verifies this expression directly. The branch connected to zero heating has $0 < c \leq 1$, equivalently

$$c = \frac{\text{Ng}}{1 - \text{Ng} + \sqrt{1 - 2\text{Ng}}}, \quad 0 < \text{Ng} \leq \frac{1}{2}. \quad (35)$$

Integrating Eq. (18) gives $M = 1 + c$. Differentiation of $2c/(1+c)^2$ identifies the exact turning point

$$\text{Ng}_{\text{turn}} = \frac{1}{2}, \quad \theta_{c,\text{turn}} = 2 \log 2. \quad (36)$$

The second branch has $c > 1$. These exact relations provide nonlinear checks of temperature, flow rate, and continuation without relying on a second implementation of the same discretization.

At $N = 32$, the five nonlinear Newtonian checks span $\text{Ng} = 0.05$ to 0.49 ; their maximum nodal temperature error is below 10^{-8} . Table 2 and Figure 1 examine $\text{Ng} = 0.45$. The error decreases rapidly from $N = 8$ to 16 and then reaches a floating-point and nonlinear-solver floor. Differentiated off-grid residuals are more sensitive to numerical cancellation and need not decrease monotonically once this floor is reached. Reporting that floor prevents roundoff-level agreement from being interpreted as unlimited accuracy.

4.3. Independent Solver Comparisons

Table 3 includes a near-turning-point Newtonian case and five sPTT cases with different sensitivities. The largest temperature difference is 2.46×10^{-10} , and the largest relative Nusselt-number difference is 3.13×10^{-10} . These comparisons use different meshes, residual controls, and velocity integrations. Agreement verifies the stated reduced equations, rather than validating their applicability to a specific polymer or processing experiment.

Table 2. Nonlinear Newtonian convergence at $\text{Ng} = 0.45$. The error is measured against the exact field; the off-grid residual includes differentiated interpolation error.

N	$\ \theta_N - \theta_{\text{exact}}\ _{\infty}$	Nu	Off-grid residual
8	1.52×10^{-7}	7.815449077	1.97×10^{-5}
12	3.62×10^{-11}	7.815446748	2.92×10^{-7}
16	2.31×10^{-13}	7.815446736	1.52×10^{-10}
24	2.52×10^{-13}	7.815446736	6.26×10^{-11}
32	4.16×10^{-13}	7.815446736	1.54×10^{-9}
40	2.12×10^{-13}	7.815446736	1.87×10^{-10}

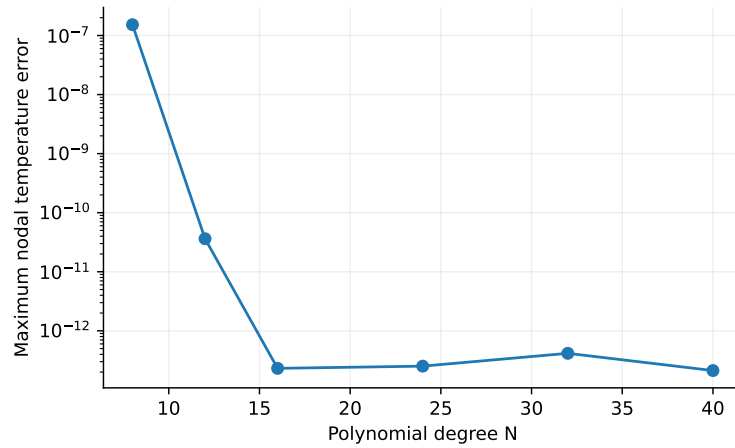


Figure 1. Convergence against Eq. (34) at $\text{Ng} = 0.45$, $\chi = 0$. The plotted error is the maximum absolute difference at the collocation points.

Table 3. Spectral-adaptive BVP comparisons. E_{θ} is the maximum absolute temperature difference, and E_{Nu} is the relative Nusselt-number difference.

Ng	χ	a	BVP nodes	E_{θ}	E_{Nu}
0.10	0.01	0.5	121	9.13×10^{-12}	2.85×10^{-11}
0.20	0.04	0.5	386	8.51×10^{-12}	9.66×10^{-12}
0.20	0.04	1.0	241	2.95×10^{-12}	8.83×10^{-13}
0.20	0.04	1.5	135	1.06×10^{-11}	1.66×10^{-11}
0.20	0.04	2.0	241	2.46×10^{-10}	3.13×10^{-10}
0.49	0.00	1.0	360	2.93×10^{-11}	2.57×10^{-12}

4.4. Energy Partition and Global Balances

The two contributions in Eq. (8) are evaluated separately for five ϕ values at $\text{Ng} = 0.15$, $\chi = 0.04$, and $a = 0.5$. Each source is used in a separate solve. Table 4 reports the resulting values and their differences from $\phi = 0$. The largest temperature discrepancy is 1.34×10^{-14} . This confirms the analytical invariance; it does not support a nonzero partition-induced heat-transfer correction in the present equilibrium.

Table 4. Computed source-partition invariance at $\text{Ng} = 0.15$, $\chi = 0.04$, $a = 0.5$. E_{ϕ} is the maximum temperature difference from the $\phi = 0$ solve.

ϕ	θ_c	M	Nu	E_{ϕ}
0.00	0.327116412	2.334203473	10.102339868	0
0.25	0.327116412	2.334203473	10.102339868	9.83×10^{-15}
0.50	0.327116412	2.334203473	10.102339868	9.94×10^{-15}
0.75	0.327116412	2.334203473	10.102339868	1.34×10^{-14}
1.00	0.327116412	2.334203473	10.102339868	7.72×10^{-15}

Across the 80-case production grid, the largest relative discrepancies in the integrated thermal balance and pressure-power balance are both 2.76×10^{-10} . The largest normalized off-grid differential residual is 1.20×10^{-9} . All validation assertions in the distributed computation script pass. The residuals, balance errors, and independent-solver differences are supplied as numerical verification measures, without a statistical or experimental uncertainty interpretation.

5. Results and Discussion

5.1. Parameter Domain and Interpretation

The production grid is the Cartesian product

$$Ng \in \{0.01, 0.05, 0.10, 0.15, 0.20\}, \quad \chi \in \{0, 0.0025, 0.01, 0.04\}, \quad a \in \{0.5, 1, 1.5, 2\}. \quad (37)$$

All 80 combinations converge on the zero-heating-connected branch. At $\chi = 0$, changing a has no effect because the relaxation-dependent term vanishes; these repeated control cases are included in the grid count. The independent values of Ng and χ describe a model-space sensitivity study. Changing pressure in a single physical device would generally change both groups, and requires following the corresponding joint parameter path.

The source structure in Eq. (15) separates a term proportional to e^θ from the sPTT contribution proportional to $e^{(3-2a)\theta}$. At $a = 0.5$, the latter grows as $e^{2\theta}$. At $a = 1$, both terms grow as e^θ . At $a = 1.5$, the sPTT contribution is independent of temperature, and at $a = 2$ it decreases as $e^{-\theta}$. This decomposition explains why the relaxation-time sensitivity can alter feedback even though viscosity always decreases with temperature.

5.2. Effect of Nahme Number and Radial Profiles

Figure 2 shows temperature and mean-normalized velocity at $\chi = 0.04$, $a = 0.5$. Increasing Ng from 0.01 to 0.20 increases the centerline temperature excess and decreases the central viscosity ratio. At $Ng = 0.20$, $\theta_c = 0.547642$, $\eta_c/\eta_w = 0.578312$, and $M = 2.741610$. The larger flow rate at prescribed G also increases the pressure-work input and, by Eq. (22), the required wall cooling.

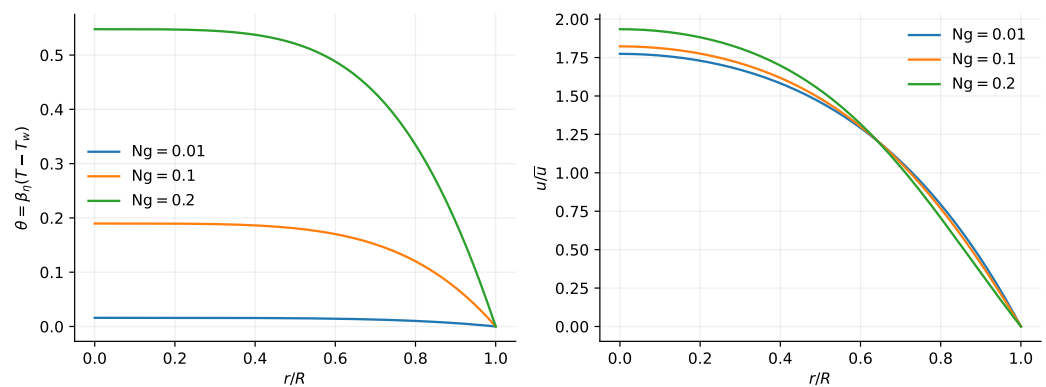


Figure 2. Radial temperature excess and mean-normalized velocity for $\chi = 0.04$, $a = 0.5$. All curves have prescribed wall temperature and pressure gradient. The normalization of velocity uses the computed mean for each curve.

Temperature is largest at the centerline and decreases toward the wall. The zero centerline shear does not imply zero centerline temperature: heat generated at larger radii is redistributed by conduction. With $S \geq 0$, integration of Eq. (17) establishes $\theta_y \leq 0$ for the regular solutions, consistent with the profiles.

The increase in M should be interpreted together with the normalization of the profiles. The dimensional wall shear rate remains fixed by Eq. (23), while the velocity divided by its

mean changes shape. A larger temperature excess therefore need not imply a larger wall shear rate under the controls used here.

5.3. Influence of Relaxation-Time Sensitivity

Figure 3 and Table 5 quantify the effect of a at $\chi = 0.04$. At $\text{Ng} = 0.20$, increasing a from 0.5 to 2 reduces θ_c from 0.547642 to 0.353380 and M from 2.741610 to 1.946714. The outward dimensionless flux decreases from 2.193288 to 1.557371, whereas Nu increases from 9.416165 to 10.534876.

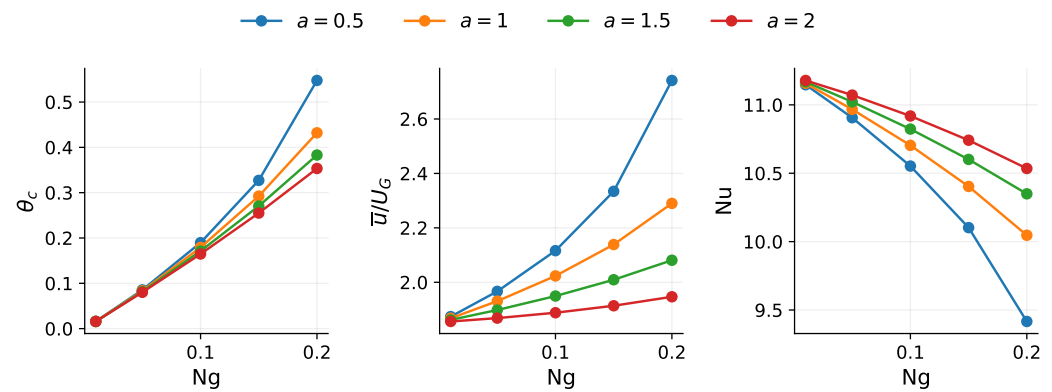


Figure 3. Temperature, mean velocity, and apparent Nusselt number versus Ng for $\chi = 0.04$. The four values of a correspond to different prescribed temperature sensitivities of relaxation time relative to viscosity.

Table 5. Representative computed wall-cooled states at $\chi = 0.04$. The wall temperature and pressure gradient are prescribed; all listed fluxes are calculated.

Ng	a	θ_c	M	q^*	Nu
0.10	0.5	0.189670	2.116163	0.846465	10.552474
0.20	0.5	0.547642	2.741610	2.193288	9.416165
0.10	1.0	0.179250	2.023456	0.809382	10.703874
0.20	1.0	0.432096	2.289896	1.831917	10.046567
0.10	1.5	0.171173	1.949432	0.779773	10.822846
0.20	1.5	0.383071	2.080969	1.664775	10.349477
0.10	2.0	0.164662	1.888146	0.755258	10.918602
0.20	2.0	0.353380	1.946714	1.557371	10.534876

The reduction in heating reflects the faster decrease of relaxation time and the resulting attenuation of the shear-dependent contribution. The rise in Nu does not mean that the absolute wall heat flux rises: Nu is a ratio involving the bulk-to-wall temperature difference. Both the heat flux and its driving temperature difference must be examined before interpreting changes as an engineering improvement.

For every sampled a at $\chi = 0.04$, Nu decreases with increasing Ng over the tested interval. For example, at $a = 0.5$ it decreases from 10.552474 at $\text{Ng} = 0.10$ to 9.416165 at $\text{Ng} = 0.20$. This trend is a computed result for the present fixed-pressure, wall-cooled state. It should not be transferred unchanged to prescribed-flow-rate or wall-heating problems.

5.4. Influence of the sPTT Parameter

In the constant-property auxiliary limit, increasing χ from zero to 0.04 raises Nu_0 from 9.6 to 11.203150 and increases M_0 from one to 1.853333. Nonlinear thermal feedback modifies both quantities through Eq. (15). The relaxation and extensibility parameters enter the velocity and temperature equations through χ , so those fields alone cannot identify ϵ and We_G separately. The normal stress retains an explicit factor of We_G in Eq. (13).

This distinction matters when interpreting a parameter sweep as a material characterization. The calculations determine the thermal consequences of specified model coefficients; they do not estimate those coefficients from data. Independent rheological measurements would be needed to separate the parameters and select a physically representative temperature range.

5.5. Steady-Branch Turning Points

Figure 4 traces Ng against prescribed centerline temperature for $\chi = 0.04$. Table 6 also includes $\chi = 0.01$ and the exact Newtonian control. The turning-point values increase with a in the tested range. At $\chi = 0.04$, they are 0.226413, 0.321372, 0.408756, and 0.452037 for $a = 0.5, 1, 1.5$, and 2, respectively. The Newtonian family gives 0.5.

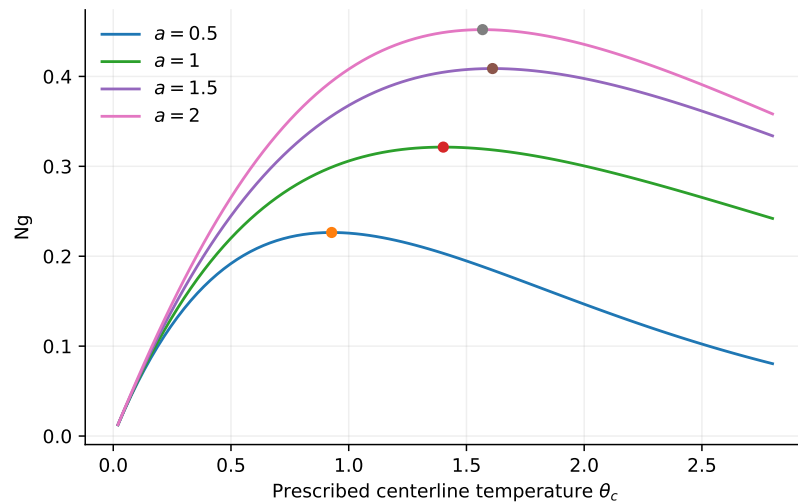


Figure 4. Centerline-parameter continuation at $\chi = 0.04$. Markers indicate the maxima of Ng on the traced branches. The descending portions are steady solutions reached by prescribing centerline temperature; no dynamic-stability classification is assigned.

Table 6. Turning points of the traced steady branches at $N = 32$. E_{BVP} is the relative difference in Ng_{turn} from the independent adaptive calculation

χ	a	Ng_{turn}	$\theta_{c,turn}$	E_{BVP}
0.00	1.0	0.500000	1.386294	9.94×10^{-11}
0.01	0.5	0.379009	1.142132	3.20×10^{-13}
0.01	1.0	0.439416	1.393050	1.12×10^{-12}
0.01	1.5	0.471838	1.451003	2.37×10^{-13}
0.01	2.0	0.485776	1.439813	2.54×10^{-10}
0.04	0.5	0.226413	0.927550	6.82×10^{-13}
0.04	1.0	0.321372	1.401592	1.11×10^{-11}
0.04	1.5	0.408756	1.610713	9.28×10^{-11}
0.04	2.0	0.452037	1.568357	2.57×10^{-12}

For $a = 0.5$, raising χ from 0.01 to 0.04 reduces the turning-point Ng from 0.379009 to 0.226413, consistent with the stronger positive temperature dependence of the sPTT heating term. Increasing a moderates that contribution and shifts the turning point toward a larger Ng .

The independent adaptive turning-point calculations differ from the spectral values by less than 2.6×10^{-10} relatively. For $\chi = 0.04$, $a = 0.5$, degrees $N = 16, 24, 32, 40$ give turning-point Ng values agreeing within 6×10^{-13} . The centerline location is less precisely determined by the scalar maximum because $Ng(A)$ is locally flat. Accordingly, the reported

θ_c values are rounded to six decimals, and the main conclusions rely on the better-resolved turning-point parameter.

A turning point describes the continuation geometry of steady equilibria. A statement about temporal thermal runaway, oscillations, or stability would require the time-dependent energy and constitutive equations and an eigenvalue or transient analysis. No such inference is made from failure of a fixed-parameter Newton solve or from the shape of Figure 4 alone.

6. Scope, Limitations, and Reproducibility

The calculations constitute numerical verification of a specified continuum model. They contain no experiments, fitted polymer parameters, or measured thermal properties. The use of one mode, a linear stress function, constant conductivity, zero solvent viscosity, zero expansion, and a steady parallel velocity limits direct process prediction. A real-fluid application requires checking laminar conditions, the validity of the exponential laws, material degradation or phase changes, and the adequacy of the selected rheological model.

The source identity depends on the absence of material evolution of the stress and temperature in the adopted geometry. Developing flow, entrance regions, secondary motion, transient heating, or nonzero material-property derivatives require revisiting the full constitutive and energy transport. The present result also does not establish a global free-energy formulation for independently chosen property laws. Its thermodynamic claim is the correctly reduced energy-source equality in the stated equilibrium.

The thermal boundary condition is prescribed temperature, and the wall flux is calculated. A constant-flux formulation must include an appropriate axial thermal balance and absolute-temperature condition. The classical negligible-dissipation constant-flux value 48/11 consequently belongs to a different problem from the 48/5 benchmark used here. Distinguishing those configurations is necessary for meaningful validation.

The dataset covers a bounded parameter region and continuation interval. It does not establish a complete phase diagram, uniqueness for all coefficients, or stability of any branch. Increasing N beyond the convergence floor can amplify floating-point differentiation defects, so analytical error and off-grid residual checks should accompany any future extension to steeper profiles.

The accompanying package supplies the numerical source, exact benchmark functions, all 80 case records, independent-solver comparisons, partition checks, continued branch coordinates, turning-point convergence, and the plotted radial profiles. A single command regenerates the data and figures. Every table is generated from those CSV records. The original preliminary tables are superseded by calculations for the explicitly defined wall-cooled configuration.

7. Conclusions

The steady linear-sPTT pipe model admits an exact local equality between mechanical stress power and relaxation heating even when viscosity and relaxation time vary radially. Consequently, the reduced energy source is independent of ϕ at fixed rheology. Five separately evaluated numerical partitions confirm this equality to a maximum nodal temperature difference of 1.34×10^{-14} .

Prescribing the wall temperature produces a closed dissipation–conduction equilibrium. Analytical stress elimination and the squared-radius coordinate yield a compact Chebyshev formulation with an explicit centerline equation. Constant-property polynomial benchmarks, an exact nonlinear Newtonian family, independent adaptive solutions, and global heat and pressure-power balances verify the implementation. The largest conservation discrepancy over the 80-case grid is 2.76×10^{-10} .

The property-sensitivity ratio controls thermal feedback through the exponent $3 - 2a$ in the sPTT source contribution. At $\chi = 0.04$, $Ng = 0.20$, increasing a from 0.5 to 2 lowers the centerline temperature and mean flow while raising the apparent Nusselt number. The turning-point Ng on the traced steady branch increases from 0.226413 to 0.452037 over

the same sensitivity range. These findings quantify steady model behaviour and identify verification targets for more general non-isothermal solvers. Their extension to transient stability and experimentally calibrated processing conditions remains a separate research task.

Nomenclature

Symbol	Definition
R, r, z	Pipe radius and radial and axial coordinates
$u, \bar{u}$	Axial and cross-sectionally averaged velocities
$G = -dp/dz > 0$	Prescribed axial pressure-gradient magnitude
T, T_w, T_b	Local, prescribed wall, and velocity-weighted bulk temperatures
η_w, λ_w	Zero-shear polymer viscosity and relaxation time at T_w
$\beta_\eta, \beta_\lambda$	Temperature-sensitivity coefficients, with units K^{-1}
k	Constant thermal conductivity
ϵ	Parameter in the linear sPTT stress function
τ, S	Polymeric extra-stress and rate-of-deformation tensors
ϕ	Parameter in the reduced energy-source partition, $0 \leq \phi \leq 1$
$U_G = GR^2/(8\eta_w)$	Reference Newtonian mean velocity for the prescribed pressure gradient
$x = r/R, y = x^2$	Dimensionless radius and squared-radius coordinate
$\theta = \beta_\eta(T - T_w)$	Dimensionless temperature excess
$v = u/U_G, M = \bar{u}/U_G$	Dimensionless local and mean velocities
$We_G = \lambda_w U_G / R$	Pressure-scaled Weissenberg number
$\chi = \epsilon We_G^2$	Combined parameter governing the present flow and thermal fields
$Ng = \beta_\eta \eta_w U_G^2 / k$	Pressure-scaled Nahme number
$a = \beta_\lambda / \beta_\eta$	Ratio of property-sensitivity coefficients
$q_w, q^* = \beta_\eta R q_w / k$	Outward wall heat flux and its dimensionless form
$Nu = 2Rq_w / [k(T_b - T_w)]$	Diameter-based apparent Nusselt number for the present equilibrium
N	Degree of the temperature interpolating polynomial

Author Contributions

Both authors contributed equally in this research.

Competing Interests

The authors declare that there are no competing interests.

Data and Code Availability

The computational source, machine-readable results, figure data, and validation summary are supplied with this manuscript. The calculations were executed with Python 3.12.14, NumPy 2.3.5, SciPy 1.17.0, and Matplotlib 3.10.8. No external experimental dataset is required.

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Use of Generative Artificial Intelligence

Generative artificial intelligence tools were used solely for language editing, grammatical correction, and improvement of readability. The author reviewed and verified the final manuscript and assumes full responsibility for its accuracy, integrity, and scholarly content.

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Appendix A. Independent Derivation of the Exact Benchmarks

For frozen properties, $S = 16y + 512\chi y^2$. The operator $4(y\partial_{yy} + \partial_y)$ maps $1 - y^2$ to $-16y$ and $1 - y^3$ to $-36y^2$. This directly gives Eq. (32). Integrating $v_y = -2(1 + 32\chi y)$ from the wall gives Eq. (31). Multiplication of these polynomials and exact integration on $[0, 1]$ gives Eq. (33).

For the nonlinear Newtonian family, Eq. (34) gives

$$\theta_y = -\frac{4cy}{1 + cy^2}, \quad 4(y\theta_{yy} + \theta_y) = -\frac{32cy}{(1 + cy^2)^2}. \quad (\text{A1})$$

Also $16\text{Ng}y^\theta = 16\text{Ng}y(1 + c)^2/(1 + cy^2)^2$. Their cancellation requires $\text{Ng} = 2c/(1 + c)^2$. Define

$$F_c(y) = \frac{y}{2(1 + cy^2)} + \frac{\arctan(\sqrt{c}y)}{2\sqrt{c}}. \quad (\text{A2})$$

Since $F'_c(y) = (1 + cy^2)^{-2}$, the exact velocity is

$$v(y) = 2(1 + c)^2 [F_c(1) - F_c(y)]. \quad (\text{A3})$$

Exchanging the order of integration gives

$$M = 2(1 + c)^2 \int_0^1 \frac{y}{(1 + cy^2)^2} dy = 1 + c. \quad (\text{A4})$$

These derivations provide direct substitution tests of the equations and independent exact values for the validation script.

Appendix B. Reproduction Procedure

From the package root, execute the following commands:

```
python3 code/run_study.py
python3 code/make_tables.py
```

They regenerate the CSV files, verification summary, figure PDFs and PNGs, and table fragments. Compile `main.tex` twice with PDFLaTeX. The Python dependency versions used for the reported run are listed in `requirements.txt`. The computation script stops on failed benchmark, conservation, independent-solver, or partition assertions. Automated checks are acceptance criteria for the reported model calculations and do not replace scientific assessment of the assumptions.