Impulse-Momentum Balance Approach for Planar Flexible Multibody Systems with Frictional Impacts

William W. Predebon[1,*]

[1] Retired professor and chair of the Department of Mechanical Engineering-Engineering Mechanics, at Michigan Technological University
* Correspondence: w.prebon@gmail.com

Abstract: Traditional approaches to modeling normal impacts in rigid multibody systems rely on momentum balance equations, often invoking Newton's hypothesis to define the restitution coefficient via relative normal velocities before and after impact. However, when friction is involved, the impact dynamics become more intricate, allowing for various modes such as sliding, sticking, or reverse sliding. The momentum balance equations involve changes in velocity and two impulse components - normal and tangential to the contact surfaces. To solve these equations, two additional conditions are required: one from Coulomb's law and the other from the restitution coefficient definition. Unfortunately, Newton's hypothesis yields inaccurate results, necessitating the use of Poisson's hypothesis, which defines the restitution coefficient through normal impulses during compression and restitution periods. This paper presents a novel formulation for impacts with friction in planar flexible multibody systems. We employ the floating frame of reference formulation to model flexible bodies and develop a computational algorithm, leveraging Routh's graphical techniques, to calculate normal and tangential impulses at the contact point.

Keywords: flexible multibody dynamics, impact with friction, momentum balance equations, routh's diagram, poisson's hypothesis

Citation: Predebon, W. W. 2020. Impulse-Momentum Balance Approach for Planar Flexible Multibody Systems with Frictional Impacts . *TK Techforum Journal (ThyssenKrupp Techforum)* 2020(1): 1–7.

Received: June-18-2019
Accepted: December-22-2019
Published: June-30-2020

1. Introduction

The development of the generalized impulse-momentum balance equations stemmed from the application of impulsive dynamics to collisions in multibody systems. Initially, the impulse-momentum model was successfully applied to rigid body multibody systems [1,2]. Later, it was extended to include both rigid and flexible bodies through algebraic equations [1]. This extension incorporated flexibility via the finite element method and utilized component mode synthesis to reduce flexible coordinates, with Newton's rule defining the coefficient of restitution.

However, the impulse-momentum model was initially based on rigid body considerations, necessitating examination of its applicability to impacts involving deformable bodies. Unlike rigid bodies, deformable bodies do not experience instantaneous velocity jumps throughout the body upon impact, except at the point of contact. Several studies [3] investigated the applicability of the rigid body concept of the coefficient of restitution to flexible bodies, concluding that the momentum balance approach can confidently predict impact responses in flexible body dynamics.

Further research [4] demonstrated that the impulse-momentum model yields consistent results when using a sufficient number of deformation modes, regardless of the selected mode shapes. The velocity of every point on the flexible body converges to zero, except in the impact zone, and there is no jump in reaction forces immediately after impact. This indicates that the impulsive force propagates as a wave with a finite speed.

Previous works solved only a simple balance, artificially removing the impacting rigid body from the contact area after the first balance. However, impacts involving flexible bodies have a finite duration, generating elastic waves that excite the flexible body's geometry and are reflected back multiple times before the process concludes. The impulse-momentum model's assumption of constant coordinates contrasts with the analysis of impact-induced vibrations, which arise from flexible bodies' elastic motion during contact. Despite this, the generalized impulse-momentum balance equations remain effective [2].

The impulse-momentum equations remain valid due to the stepwise solution of the collision process, which involves simulating the impact through multiple balances. This numerical approach allows the system configuration to change between balances, unrelated to the actual phenomenon of successive impacts. The number of times the generalized impulse-momentum balance equations must be solved to complete the impact process depends on the number of coordinates used to describe flexibility and the time step employed.

As the number of flexible degrees of freedom increases, so does the number of balances required, resulting in a decrease in the intensity of instantaneous impacts. This occurs because the mass associated with the contact section decreases as the geometric discretization is refined. In a continuous approach, the mass section is zero, allowing its velocity to change instantaneously with zero impulse [5].

The coefficient of restitution, included in the formulation to account for energy losses near the contact area, becomes challenging to interpret under these conditions. This is understandable, as continuous contact is simulated as a virtual succession of instantaneous impacts, where energy is assumed to be lost locally at each impact. Despite this, satisfactory results are achieved with a unity coefficient of restitution.

The previous works mentioned earlier did not consider friction, which adds complexity to the problem. Friction at contact points or surfaces can lead to various impact modes, such as sticking, sliding, or reverse sliding. The generalized impulse-momentum balance equations must account for velocity changes and two impulse components: one in the normal direction and the other in the tangential direction of the impacting surfaces. To solve these equations, two additional conditions are required: one from Coulomb's law of friction and the other from the definition of the coefficient of restitution.

Several authors (Brach, 1989; Kane, 1984) have demonstrated that using Newton's hypothesis can violate energy conservation principles in certain cases. Therefore, Poisson's rule is generally preferred for frictional impacts. [1] studied frictional impact in planar rigid multibody systems, developing an algorithm to generate Routh's diagrams (Routh, 1891) for calculating impulse components. [3] used similar equations but employed the linear complementary problem technique to calculate impulse components, which is computationally efficient but slightly deviates from Coulomb's law.

[5] compared both formulations and found that they yield similar results in most cases, but differ in specific situations. These differences occur when reverse sliding follows sliding or when sticking is followed by sliding, highlighting the need for careful consideration of frictional effects in impact dynamics.

This paper builds upon the formulation proposed [4] and extends it to the frictional impact of planar flexible multibody systems. The floating frame of reference is utilized to describe flexibility, and Poisson's hypothesis is employed to define the coefficient of restitution. Routh's diagrams are used to calculate the impulse components, and the continuous impact of finite duration is simulated through successive virtual infinitesimal impacts.

The motivation behind this work is to investigate whether the presence of friction affects the applicability of the impulse-momentum approach, originally developed for rigid bodies, to systems with flexible bodies. If successful, this approach will provide an alternative to established force-based methods.

2. Floating Frame of Reference Formulation

The floating reference methods are widely used in the literature, with numerous authors contributing to their development. To describe the system's movement, a coordinate system is associated with each body, capturing the rigid body motion. Elastic displacements, calculated using small deformation theory, are superimposed onto the rigid motion. The global position of an arbitrary point on body i can be expressed as:

$$r_j = R_j + A_j \bar{u}_{oj} + S_j q_{fj} \tag{1}$$

where:

- R_j is the set of Cartesian coordinates defining the body reference origin's location
- A_j is the transformation matrix between local and global reference systems
- $\bar{u}_{oj}$ is the point's position in the undeformed state, expressed in the local system
- S_j is a space-dependent shape matrix
- q_{fj} is the vector of time-dependent elastic generalized coordinates of the deformable body

This separation between rigid body motion and elastic displacements is not unique, allowing for different reference conditions to be selected for the same problem. The analyst should choose the best reference conditions, typically those yielding the smallest number of elastic coordinates. The main advantage of this representation is its ability to efficiently describe complex motions.

3. Applicability of the Component Synthesis Method

The component synthesis method is highly applicable, enabling a significant reduction in the number of elastic coordinates. This method is computationally efficient, making it a valuable tool. Typically, the deformed shape of the body is represented by superimposing the normal vibration modes of the body, constrained by the reference conditions. In most cases, static deformation modes are not necessary, although they can occasionally improve convergence or simplify the formulation, such as in natural coordinates with fixed frontiers, where they facilitate imposing kinematical constraints by sharing coordinates.

The Lagrange multipliers technique is employed to account for coordinate constraints. The equations of motion are:

$$\begin{aligned} M\ddot{q} + Kq + C^T\lambda &= Q \\ Cq - t &= 0 \end{aligned} \tag{2}$$

where:

- M is the mass matrix
- K is the stiffness matrix
- C is the constraint matrix
- q is the vector of generalized coordinates
- λ is the vector of Lagrange multipliers
- Q is the vector of external forces
- t is the vector of constraint forces

4. Generalized Impulse-Momentum Balance Equations

Assuming a very short time interval for impact, the reference and flexible coordinates are considered constant. Integrating the equations of motion during this interval yields:

$$\begin{aligned} M\Delta\dot{q} + C^T\lambda &= P_g \\ C_q\Delta\dot{q} &= 0 \end{aligned} \tag{3}$$

Eliminating λ from these equations gives:

$$\lambda = [C_q M^{-1} C^T]^{-1} C_q M^{-1} P_g = HP_g \tag{4}$$

Substituting Eq. (4) into Eq. (3) results in:

$$\Delta\dot{q} = M^{-1}[I - C_q^T H]P_g \tag{5}$$

Since the velocity problem in a multibody system is linear, the relative velocity of contact points can be expressed as a function of the derivative of the coordinates vector:

$$v_r = D\dot{q} \tag{6}$$

The matrix D, dependent on the position at each time step, is used to obtain the tangential and normal components of the relative velocity. Two unit vectors, n and t, are defined, with n perpendicular to the contacting surfaces and t along the tangential direction and perpendicular to n. The relative velocity components can be expressed as:

$$\begin{aligned} v_n &= n^T v_r = n^T D\dot{q} = c_n^T \dot{q} \\ v_t &= t^T v_r = t^T D\dot{q} = c_t^T \dot{q} \end{aligned} \tag{7}$$

Using Eqs. (5) and (7), we can write:

$$\begin{aligned} v_n^+ &= v_n^- + \Delta v_n = v_n^- + c_n^T \Delta\dot{q} = v_n^- + c_n^T M^{-1}[I - C_q^T H]P_g \\ v_t^+ &= v_t^- + \Delta v_t = v_t^- + c_t^T \Delta\dot{q} = v_t^- + c_t^T M^{-1}[I - C_q^T H]P_g \end{aligned} \tag{8}$$

where v_n^+ and v_n^- are the relative normal velocities after and before impact, respectively, and similarly for the relative tangential velocities. The generalized impulse can be separated into normal and tangential components:

$$P_g = \int (f_n n + f_t t)dt = c_n \int f_n dt + c_t \int f_t dt = c_n P_n + c_t P_t \tag{9}$$

where f_n is the normal contact force, f_t is the frictional force, and P_n and P_t are the normal and tangential impulses due to impulsive forces f_n and f_t. Substituting Eq. (9) into Eq. (8) yields:

$$\begin{aligned} v_n^+ &= v_n^- + m_{nn} P_n + m_{nt} P_t \\ v_t^+ &= v_t^- + m_{nt} P_n + m_{tt} P_t \end{aligned} \tag{10}$$

Assuming a very short time interval for impact, the reference and flexible coordinates are considered constant. Integrating the equations of motion during this interval yields:

$$\begin{aligned} M\Delta\dot{q} + C^T\lambda &= P_g \\ C_q \Delta\dot{q} &= 0 \end{aligned} \tag{11}$$

Eliminating λ from these equations gives:

$$\lambda = [C_q M^{-1} C^T]^{-1} C_q M^{-1} P_g = H P_g \tag{12}$$

Substituting Eq. (4) into Eq. (3) results in:

$$\Delta\dot{q} = M^{-1}[I - C_q^T H]P_g \tag{13}$$

Since the velocity problem in a multibody system is linear, the relative velocity of contact points can be expressed as a function of the derivative of the coordinates vector:

$$v_r = D\dot{q} \tag{14}$$

The matrix D, dependent on the position at each time step, is used to obtain the tangential and normal components of the relative velocity. Two unit vectors, n and t, are defined, with n perpendicular to the contacting surfaces and t along the tangential direction and perpendicular to n. The relative velocity components can be expressed as:

$$\begin{aligned} v_n &= n^T v_r = n^T D\dot{q} = c_n^T \dot{q} \\ v_t &= t^T v_r = t^T D\dot{q} = c_t^T \dot{q} \end{aligned} \tag{15}$$

Using Eqs. (5) and (7), we can write:

$$\begin{aligned} v_n^+ &= v_n^- + \Delta v_n = v_n^- + c_n^T \Delta \dot{q} = v_n^- + c_n^T M^{-1}[I - C_q^T H] P_g \\ v_t^+ &= v_t^- + \Delta v_t = v_t^- + c_t^T \Delta \dot{q} = v_t^- + c_t^T M^{-1}[I - C_q^T H] P_g \end{aligned} \tag{16}$$

where v_n^+ and v_n^- are the relative normal velocities after and before impact, respectively, and similarly for the relative tangential velocities. The generalized impulse can be separated into normal and tangential components:

$$P_g = \int (f_n n + f_t t) dt = c_n \int f_n dt + c_t \int f_t dt = c_n P_n + c_t P_t \tag{17}$$

where f_n is the normal contact force, f_t is the frictional force, and P_n and P_t are the normal and tangential impulses due to impulsive forces f_n and f_t. Substituting Eq. (9) into Eq. (8) yields:

$$\begin{aligned} v_n^+ &= v_n^- + m_{nn} P_n + m_{nt} P_t \\ v_t^+ &= v_t^- + m_{nt} P_n + m_{tt} P_t \end{aligned} \tag{18}$$

where the generalized impact coefficients m_{nn}, m_{nt}, and m_{tt} are defined as:

$$\begin{aligned} m_{nn} &= c_n^T M^{-1}[I - C_q^T H] c_n \\ m_{nt} &= c_n^T M^{-1}[I - C_q^T H] c_t \\ m_{tt} &= c_t^T M^{-1}[I - C_q^T H] c_t \end{aligned} \tag{19}$$

These coefficients can be evaluated knowing the inertia properties and coordinates at the time of impact. Once evaluated, only the normal and tangential impulses P_n and P_t are needed to determine the velocities after impact, as shown in Eq. (10).

5. Routh's Diagrams

Routh's method combines Poisson's rule and Coulomb's friction law with Eq. (10) to calculate the generalized impulses. Poisson's hypothesis defines the coefficient of restitution as:

$$e = \frac{P_{nR}}{P_{nC}} \tag{20}$$

relating the accumulated normal impulses during compression (P_{nC}) and restitution periods (P_{nR}). Therefore:

$$P_n = P_{nC} + P_{nR} = (1 + e) P_{nC} \tag{21}$$

Coulomb's law provides the sliding and sticking conditions, which can be described as:

$$\text{sticking: } dP_t < \mu dP_n \tag{22}$$

where μ is the friction coefficient. [3] developed a graphical technique to determine impulses, using diagrams that represent normal impulse versus tangential impulse. These diagrams feature four distinct lines, each corresponding to different contact point scenarios:

- Three lines represent possible contact point behaviors: sliding, sticking, or reverse sliding.
- The fourth line represents the maximum compression point, marking the end of the compression period when reached.

These lines can be mathematically expressed as:

$$
\begin{aligned}
\text{Line of limiting friction (F): } & P_t = -\mu\left(\frac{v_t^-}{|v_t^-|}\right)P_n \\
\text{Line of sticking (S): } & v_t^- + m_{nt}P_n + m_{tt}P_t = 0 \\
\text{Line of maximum compression (C): } & v_n^- + m_{nn}P_n + m_{nt}P_t = 0
\end{aligned} \tag{23}
$$

These equations define the boundaries for different contact scenarios, enabling the determination of impulses and subsequent velocities after impact.

6. Summary of the Procedure

When contact is detected, the integration of the equations of motion (Eq. 2) is halted. Knowing the position and velocity coordinates at the time of impact (q and $\dot{q}^-$), the velocities after impact ($\dot{q}^+$) are calculated using the following steps:

1. Evaluate vectors c_n and c_t
2. Evaluate matrix H
3. Evaluate generalized impact parameters m_{nn}, m_{nt}, and m_{tt} (Eq. 11)
4. Evaluate relative normal and tangential velocities before impact (v_n^- and v_t^-)
5. Plot Routh's diagram (Eqs. 15, 16, and 17) and evaluate P_{nC} and P_{tC}
6. Evaluate total accumulated normal impulse P_n (Eq. 13)
7. Evaluate total accumulated tangential impulse P_t from Routh's diagram
8. Evaluate change in system velocities $\Delta\dot{q}$ (Eqs. 5 and 9)
9. Evaluate velocities after impact: $\dot{q}^+ = \dot{q}^+ \delta\dot{q}$
10. Resume integration of equations of motion with updated velocities $\dot{q}^+$

Note that for simulating impact involving flexible bodies, these steps must be repeated several times before the process is complete. The first time, integration is halted, and steps (a) to (j) are applied. However, because flexible bodies may undergo multiple impacts, the process is repeated until the impact is fully resolved.

Due to the small mass associated with the contact zone of a flexible body (ideally zero), the impulse and resulting velocity change are also small. Consequently, the bodies will come into contact again after a few time steps, requiring another momentum balance solution. After the initial balance, only the contact section should experience a velocity change as the wave propagation begins. However, due to finite discretization, a larger velocity jump is observed closer to the contact zone.

Increasing the number of flexible coordinates used to model the flexible body decreases the mass associated with each degree of freedom and the contact section, resulting in:

- Smaller impulses associated with contact forces
- Velocity jumps affecting areas closer to the contact zone
- More balances needed to simulate the complete process
- Less severe impacts as flexible degrees of freedom increase

7. Numerical Example

The system is analyzed: a pendulum falling under gravity and impacting a fixed surface. The pendulum consists of a uniform rod with mass $m = 1$ kg and length $l = 1$ m, under gravity $g = 10$ m/s^2. Initially at rest in a horizontal position ($\theta = 90^\circ$), the pendulum impacts the surface when $\theta = 80^\circ$. Impact parameters are $\mu = 1$ and $e = 1$. This system, considered rigid, has been previously analyzed (Pfeiffer and Glocker, 1996), showing dissipative behavior due to reversed sliding, where the frictional impulse acts in different directions during compression and expansion.

8. Conclusion

In this paper, a momentum balance approach has been successfully applied to simulate the impact of a flexible pendulum with a fixed surface, incorporating friction and flexibility. The use of Routh's diagram and the momentum balance technique allowed for the accurate capture of the complex dynamics during the impact, including the excitation of flexible modes and the propagation of waves along the beam. The results showed that increasing the number of flexible coordinates improved the accuracy of the simulation, with the five-mode model providing a more realistic representation of the system's behavior. The approach demonstrated its ability to reproduce the expected behavior of flexible multibody systems with friction, including the induction of waves and the influence of flexibility on the impact dynamics. Future work will focus on extending this approach to more complex systems and exploring its potential applications in various fields.

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